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THE FINITE-DIMENSIONAL P_λ SPACES WITH SMALL λ

BY

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ABSTRACT

It is proved that there is a positive function $\phi(\varepsilon)$ defined for sufficiently small $\varepsilon > 0$ such that $\lim_{\varepsilon\to 0}\phi(\varepsilon) = 0$ and for every integer k and every k-dimensional $P_{1+\varepsilon}$ space E, $d(E, l_\infty^k) < 1 + \phi(\varepsilon)$.

The purpose of this paper is to continue the investigation begun in [1]. Using the ideas and the notation of [1] we will prove the following:

THEOREM 1. *There exists a positive function $\phi(\varepsilon)$ defined for sufficiently small $\varepsilon > 0$ with $\lim_{\varepsilon\to 0}\phi(\varepsilon) = 0$ such that for every integer k and every k-dimensional $P_{1+\varepsilon}$ space E, $d(E, l_\infty^k) < 1 + \phi(\varepsilon)$.*

Recall that E is called a P_γ space if whenever $F \supset E$ there is a projection P of F onto E with $\|P\| < \gamma$. We denote by $d(E, X)$ the Banach–Mazur distance between the spaces E and X.

As explained in [1] (see the introduction), in order to prove Theorem 1 it suffices to prove the following.

THEOREM 2. *There exists a positive function $\nu(\varepsilon)$ defined for sufficiently small $\varepsilon > 0$ with $\lim_{\varepsilon\to 0}\nu(\varepsilon) = 0$ such that for any positive integer n if there is a projection P of l_1^n onto a k-dimensional subspace E with $\|P\| < 1 + \varepsilon$, then $d(E, l_1^k) < 1 + \nu(\varepsilon)$.*

The following partial result, proved in [1], will be our main tool.

PROPOSITION. *There exists a positive function $\psi(\alpha)$ defined for sufficiently small $\alpha > 0$ with $\lim_{\alpha\to 0}\psi(\alpha) = 0$ such that for every n, if R is a projection of l_1^n onto a k-dimensional subspace F with $\|R\| \leqq 1 + \alpha$, then the following holds:*

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(*) *There is a subset* $A \subset \{1,2,\cdots,n\}$, *there exist an integer* $h > k(1-\psi(\alpha))$ *and* h *pairwise disjoint subsets* $A_1, A_2, \cdots, A_h$ *of* A *and there are* $i(1)$, $i(2), \cdots$, $i(h)$ *in* A *such that* $\| Re_{i(j)|A_j} \| \geqq 1-\psi(\alpha)$. *Moreover, for each* $q \in A$, $\| Re_{q|A} \| \geqq 1-\psi(\alpha)$, *and there is a* j, $1 \leqq j \leqq h$, *with* $\| Re_q - Re_{i(j)} \| \leqq \psi(\alpha)$.

We use here and below the following notation: if $x = \Sigma_{i=1}^n a_i e_i \in l_1^n$ and $C \subset \{1,2,\cdots,n\}$ then $x_{|C} = \Sigma_{i \in C} a_i e_i$.

We will prove Theorem 2 by iterating the process described in the above Proposition. In order to be able to do that without losing control over the constants we will need the following three lemmas. We assume here that P is a projection on l_1^n with $\| P \| < 1 + \varepsilon$ and that P is represented by the matrix (a_{ij}), i.e., $Pe_i = \Sigma_{j=1}^n a_{ji} e_j$ for all unit vectors $\{e_i\}_{i=1}^n$.

In the sequel we will define the constants $\delta = \delta(\varepsilon)$, $1-\mu = 1-\mu(\varepsilon)$, $\eta = \eta(\varepsilon)$, $\lambda = \lambda(\varepsilon)$ and $\beta = \beta(\varepsilon)$. These constants are all functions of ε tending to 0 with ε.

LEMMA 1. *Let* $G \subset \{1,2,\cdots,n\}$, *let* $H = \{i : \| Pe_{i|G^c} \| > 1-\delta\}$ *where* $\delta = \varepsilon^{1/2}$ *and put* $\mu = (1+2\delta)^{-1}$. *Let* $B = B(E) =$ *the unit ball of* E *and let* C *denote the convex hull of* $\{\pm Pe_i\}_{i \in G \cup H}$. *Then* $B \subset \mu^{-1} C$. *In particular, if* $\operatorname{span}\{Pe_i\}_{i \in G} \neq E$ *then* $H \neq \emptyset$.

PROOF. Let $\phi \in E^*$ satisfy the inequality $|\phi(Pe_i)| \leqq \mu$ for all $i \in G \cup H$. Let $M = \max_{1 \leqq j \leqq n} |\phi(Pe_j)| = |\phi(Pe_q)|$. Then, since $Pe_q = \Sigma_{j=1}^n a_{jq} e_j = \Sigma_{j=1}^n a_{jq} Pe_j$ we get that

$$M \leqq \mu \sum_{j \in G \cup H} |a_{jq}| + M \sum_{j \notin G \cup H} |a_{jq}| = M \sum_{j \notin G \cup H} |a_{jq}| + \mu\Big(1 + \varepsilon - \sum_{j \notin G \cup H} |a_{jq}|\Big).$$

If $M > \mu$ then, clearly, $q \notin G \cup H$ hence $\| Pe_{q|G^c} \| = \Sigma_{j \in G^c} |a_{jq}| \leqq 1-\delta$ and consequently, putting $a = \Sigma_{j \notin G \cup H} |a_{jq}|$, we get that

$$M \leqq \mu(1-a)^{-1}(1+\varepsilon-a) \leqq \mu\delta^{-1}(\delta+\varepsilon) < 1.$$

It follows that $\| \phi \| < 1$ and hence, by the Hahn–Banach theorem, $B(E) \subset \mu^{-1} C$ as claimed.

We will need the following precise description of a subspace of l_1^n which is "almost" invariant under P.

DEFINITION. Let $K \subset \{1,2,\cdots,n\}$, put $X(K) = \operatorname{span}\{e_i\}_{i \in K}$ and let $0 < \lambda < 1$. The subspace $X(K)$ will be called λ-invariant if for every $i \in K$, $\| Pe_{i|K} - Pe_i \| \leqq \lambda$.

The purpose of the next lemma is to prove the existence of λ-invariant subspaces supported outside certain sets.

LEMMA 2. *Let G, H, δ and μ be as in Lemma 1. Assume that $\|Pe_{i|_{G^c}}\| \leqq \frac{1}{2}$ for all $i \in G$ and let $\eta = \delta^{1/2} = \varepsilon^{1/4}$, $K = \{i : \|Pe_{i|_{G^c}}\| \geqq 1 - \eta\}$ and $\lambda = 10\mu^{-1}\eta$. Then $X(K)$ is a λ-invariant subspace.*

PROOF. It follows from Lemma 1 that for each $1 \leqq i \leqq n$, $Pe_i = \alpha g + \beta h$ where $g = \sum_{j\in G} a_j Pe_j$, $h = \sum_{j\in H} b_j Pe_j$, $\sum_{j\in G}|a_j| = 1 = \sum_{j\in H}|b_j|$ and $|\alpha| + |\beta| \leqq \mu^{-1}(1+\varepsilon)$. If, in particular, $i \in K$ then

$$1-\eta \leqq \|Pe_{i|_{G^c}}\| \leqq |\alpha| \sum_{j\in G} |a_j| \|Pe_{j|_{G^c}}\| + |\beta| \Big(\sum_{j\in H} |b_j|\Big)(1+\varepsilon)$$

$$\leqq \tfrac{1}{2}|\alpha| + (1+\varepsilon)|\beta| \leqq \tfrac{1}{2}|\alpha| + (1+\varepsilon)(\mu^{-1}(1+\varepsilon) - |\alpha|)$$

$$= (1+\varepsilon)^2\mu^{-1} - |\alpha|(\tfrac{1}{2}+\varepsilon)$$

hence $|\alpha| \leqq 2((1+\varepsilon)^2\mu^{-1} + \eta - 1) \leqq 3\eta$ if ε is small enough. It follows that for each $i \in K$ there is an element $h_i = \sum_{j\in H} b_{ji} Pe_j$ such that

$$\sum_{j\in H} |b_{ji}| \leqq \mu^{-1} \quad \text{and} \quad \|Pe_i - h_i\| \leqq 4\eta. \tag{1}$$

Consider now an element $Pe_j = \sum_{k=1}^{n} a_{kj} e_k$ where $j \in H$. It follows from the definition of H that $\sum_{k\in G}|a_{kj}| < \delta + \varepsilon < \eta$.

We claim that if ε is small enough then

$$\sum_{k\in K^c} |a_{kj}| \leqq 4\eta. \tag{2}$$

To prove (2) it suffices to show that $\sum_{k\in G^c - K}|a_{k,j}| \leqq 3\eta$. But for $k \in G^c - K$ we have that $\|Pe_{k|_{G^c}}\| \leqq 1 + \varepsilon - \eta$ and therefore

$$1 - \delta \leqq \|Pe_{j|_{G^c}}\| = \Big\| \sum_{k=1}^{n} a_{kj} (Pe_{k|_{G^c}}) \Big\|$$

$$\leqq (1+\varepsilon) \sum_{k\in G} |a_{kj}| + (1+\varepsilon-\eta) \sum_{k\in G^c - k} |a_{kj}| + (1+\varepsilon) \sum_{k\in K} |a_{kj}|$$

$$\leqq (1+\varepsilon)(\delta+\varepsilon) + (1+\varepsilon-\eta) \sum_{k\in G^c - K} |a_{kj}| + (1+\varepsilon)\Big(1+\varepsilon - \sum_{k\in G^c-K} |a_{kj}|\Big).$$

It follows that

$$\sum_{k\in G^c - K} |a_{kj}| \leqq \eta^{-1}((1+\varepsilon)^2 + (1+\varepsilon)(\delta+\varepsilon) - 1 + \delta) \leqq 3\eta^{-1}\delta = 3\eta$$

if $\varepsilon < 20^{-1}$ and (2) is thus proved.

Let T be the operator defined on l_1^n by $Tx = x_{|K}$. Then for each $i \in K$ we have, by (1) and (2), that

$$\| TPe_i - Pe_i \| \leqq 6\eta + \| Th_i - h_i \| \leqq 6\eta + \left\| \sum_{j \in H} b_{ji} (TPe_j - Pe_j) \right\|$$

$$\leqq 6\eta + \mu^{-1} \max_{j \in H} \| TPe_j - Pe_j \| = 6\eta + \mu^{-1} \max_{j \in H} \sum_{k \in K^c} | a_{kj} | \leqq 10\mu^{-1}\eta = \lambda.$$

It follows that $X(K)$ is a λ-invariant subspace. This proves Lemma 2.

Lemma 3. *Let X be a Banach space, let X_0 be a subspace of X and let T be a projection of X onto X_0 with $\| T \| \leqq 1 + \gamma$, where $\gamma < 1/20$. Let P be a projection on X with $\| P \| \leqq 1 + \gamma$ such that $\| TPx - Px \| \leqq \gamma \| x \|$ for all $x \in X_0$. Then there is a projection Q on X_0 such that $\| Q \| < 1 + 20\gamma$ and $\| Qx - Px \| \leqq 20\gamma$ for all $x \in B(X_0)$*

Proof. Define the operator S on X_0 by $Sx = TPx + (I - P)x$ for all $x \in X_0$. Then clearly $\| Sx - x \| \leqq \gamma \| x \|$ and hence S is an invertible operator from X_0 onto a space $Y \subset X$ with $\| S \| \leqq 1 + \gamma$ and $\| S^{-1} \| \leqq (1 - \gamma)^{-1}$. Let U be the operator defined on X by $Ux = STx + (I - T)x$. Then $\| Ux - x \| \leqq \| STx - Tx \| \leqq (1 + \gamma)\gamma \| x \|$ and therefore $\| U \| \leqq 1 + 2\gamma$, $\| U^{-1} \| \leqq (1 - 2\gamma)^{-1}$ and $\| U^{-1} - I \| \leqq 2\gamma(1 - 2\gamma)^{-1}$. Since U is one to one and $Ux = Sx$ for every $x \in X_0$ we have that U^{-1} maps Y onto X_0 hence $Q_0 = UPU^{-1}_{|Y}$ is a projection of Y onto its subspace $Z = TP(X_0)$ which happens to be also a subspace of X_0.

Also $\| Q_0 \| \leqq (1 + 2\gamma)(1 + \gamma)(1 - 2\gamma)^{-1} \leqq 1 + 8\gamma$. Note that X_0 is "close" to Y (in the sense that $\| Sx - x \| \leqq \gamma \| x \|$ for all $x \in X_0$) and there is a projection T of X onto X_0 with $\| T \| \leqq 1 + \gamma$. Hence, again, a standard perturbation argument yields the existence of a projection V of X onto Y such that $\| V \| \leqq (1 + \gamma)^2(1 - \gamma)^{-1} \leqq 1 + 4\gamma$ and $\| V - T \| \leqq 2\gamma$. Clearly $Q_0 V$ is a projection of X onto Z and hence $Q = Q_0 V_{|X_0}$ is a projection of X_0 onto Z with $\| Q \| \leqq (1 + 8\gamma)(1 + 4\gamma) < 1 + 20\gamma$ such that

$$\begin{aligned}
\| Q - P_{|X_0} \| &\leqq \| Q_0 V - PT \| = \| UPU^{-1} V - PT \| \\
&\leqq \| UPU^{-1} V - PU^{-1} V \| + \| PU^{-1} V - PU^{-1} T \| + \| PU^{-1} T - PT \| \\
&\leqq \| U - I \| \| P \| \| U^{-1} \| \| V \| + \| P \| \| U^{-1} \| \| V - T \| + \| P \| \| T \| \| U^{-1} - I \| \\
&\leqq 2\gamma(1 + \gamma)(1 + 4\gamma)^2 + (1 + \gamma)(1 + 4\gamma)2\gamma + 2(1 + \gamma)^2\gamma(1 + 4\gamma) \\
&\leqq 20\gamma.
\end{aligned}$$

This concludes the proof of Lemma 3.

A direct consequence of Lemma 3 is the following:

COROLLARY. *Let P be a projection on l_1^n with $\|P\| \leqq 1+\varepsilon$, let $\{e_i\}_{i=1}^n$ be the unit vector basis and let $K \subset \{1,2,\cdots,n\}$. Assume that $X(K) = \operatorname{span}\{e_i\}_{i\in K}$ is a λ-invariant subspace with $\lambda < 1/20$. Then there is a projection Q on $X(K)$ with $\|Q\| \leqq 1+20\lambda$ such that $\|Pe_i - Qe_i\| \leqq 20\lambda$ for all $i \in K$.*

PROOF OF THEOREM 2. We first use the Proposition with $R = P$ and $\alpha = \varepsilon$ to get a subset A (to be denoted by G_1) which satisfies (*). If $\operatorname{span}\{Pe_i\}_{i\in G_1} = E$ then, by Lemma 1, the proof is complete because (*) implies that there exist integers $i(1,1)$, $i(1,2),\cdots$, $i(1,h(1))$ in A and pairwise disjoint subsets $G_{1,1}$, $G_{1,2},\cdots$, $G_{1,h(1)}$ of G_1 such that $\|Pe_{i(1,j)}|_{G_{1,j}}\| \geqq 1-\psi(\varepsilon)$ and for each $q \in G_1$ there is a $j, 1 \leqq j \leqq h(1)$ with $\|Pe_q - Pe_{i(1,j)}\| \leqq \psi(\varepsilon)$. It follows that $\{Pe_{i(1,j)}\}_{j=1}^{h(1)}$ is a basis of E as asserted. On the other hand, if $\operatorname{span}\{Pe_i\}_{i\in G_1} \neq E$ we use Lemma 2 with $G = G_1$ and we get a subset $K = K_1$ in the complement of G_1 such that $X(K_1)$ is a λ-invariant subspace (for the projection P). The Corollary implies that there is a projection $Q = Q_1$ on the space $X(K_1)$ (which is isometrically isomorphic to $l_1^{|K_1|}$) with $\|Q_1\| \leqq 1+\beta$ where $\beta = 20\lambda$. Moreover, for each $i \in K_1$, $\|Q_1e_i - Pe_i\| \leqq \beta$. Using the Proposition with $R = Q_1$ and $\alpha = \beta$ we get a subset $A = G_2 \subset K_1$ which satisfies (*). Thus there exist integers $i(2,1)$, $i(2,2),\cdots,i(2,h(2))$ in G_2 with $h(2) > |K_1|(1-\psi(\beta))$ and these are pairwise disjoint subsets $G_{2,1}$, $G_{2,2},\cdots$, $G_{2,n}$ of G_2 such that $\|Qe_{i(2,j)}|_{G_{2,j}}\| \geqq 1-\psi(\beta)$ and for each $q \in G_2$, $\|Q_1e_q - Q_1e_{i(2,j)}\| \leqq \psi(\beta)$ for some $j, 1 \leqq j \leqq h(2)$. Since $\|Pe_i - Q_1e_i\| \leqq \beta$ for all $i \in K_1$ we get that $\|Pe_{i(2,j)}|_{G_{2,j}}\| \geqq 1-\psi(\beta)-\beta$ and for each $q \in G_2$ there is a $j, 1 \leqq j \leqq h(2)$, such that $\|Pe_q - Pe_{i(2,j)}\| \leqq \psi(\beta)+2\beta$. If $\operatorname{span}\{Pe_i\}_{i\in G_1\cup G_2} = E$ then the proof is complete. If not, we use Lemma 2 with $G = G_1 \cup G_2$ and we get a subset $K = K_2$ of the complement of $G_1 \cup G_2$ such that $X(K_2)$ is λ-invariant. On $X(K_2) \cong l_1^{|K_2|}$ we have, by the Corollary, a "good" projection Q_2 which is "close" to P. Proceeding in this manner we get after m steps $(m \leqq k)$ pairwise disjoint sets $G_1, G_2,\cdots,G_m$ such that $E = \operatorname{span}\{Pe_i\}_{i\in\cup_{k=1}^m G_k}$ and in each G_k we get pairwise disjoint subsets $G_{k,1}$, $G_{k,2},\cdots,G_{k,h(k)}$ and $h(k)$ distinct integers $i(k,1)$, $i(k,2),\cdots,i(k,h(k))$ in G_k such that $\|Pe_{i(k,j)}|_{G_{k,j}}\| \geqq 1-\psi(\beta)-\beta$ and for each $q \in G_k$ there is a $j, 1 \leqq j \leqq h(k)$, with $\|Pe_q - Pe_{i(k,j)}\| \leqq \psi(\beta)+2\beta$. Clearly the elements $\{Pe_{i(k,j)}\}_{j=1,k=1}^{h(k),m}$ form a basis of E which is "close" to a natural l_1 basis. This proves Theorem 2.

Reference

1. M. Zippin, *The range of a projection of small norm in l_1^n*, Israel J. Math. **39** (1981), 349–358.

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